Kałuża-Klein electrically charged black branes in M-theory

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Abstract

We present a class of Kałuża-Klein electrically charged black p-brane solutions of ten-dimensional, type IIA superstring theory. Uplifting to eleven dimensions these solutions are studied in the context of M-theory. They can be interpreted either as a $p+1$ extended object trapped around the eleventh dimension along which momentum is flowing or as a boost of the following backgrounds: the Schwarzschild black $(p+1)$-brane or the product of the $(10-p)$-dimensional Euclidean Schwarzschild manifold with the $(p+1)$-dimensional Minkowski spacetime.

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1 Introduction

It has been conjectured that there exists a consistent quantum theory in eleven dimensions, M-theory, whose effective field theory limit is $D = 11$ supergravity [1, 2]. Due to the presence of a 4-form field strength in this supergravity theory, there are supersymmetric electrically charged 2-brane [3] and magnetically charged 5-brane [4] solutions which are believed to play a crucial role in a precise formulation of M-theory. Double dimensional reduction of the supermembrane action [5] from eleven to ten dimensions gives rise to the type IIA superstring action [6]. This is a strong evidence for some relation between an underlying quantum theory in eleven dimensions and type IIA superstring theory. In fact, Witten [2] has argued that the strong coupling behaviour of type IIA superstring theory is given by the reduction on S^1 of $D = 11$ supergravity, with the compactification radius increasing with string coupling. This fact leads to the conjecture that compactification of M-theory on S^1 gives rise to type IIA superstring theory. The $E_8 \times E_8$ heterotic string is also believed to arrive in the M-theory reduction on $\frac{S^1}{Z_2}$ [7]. Further, the type II dualities and the heterotic-type I dualities are related to the compactification of the M-theory on T^2 and on a Z_2 orbifold of T^2, respectively [7, 8, 9]. Ten-dimensional type IIB and heterotic-type I theories at arbitrary coupling correspond to a particular point of the moduli space of the M-theory on the previous manifolds. This gives a fairly unified picture for a eleven-dimensional origin of the different superstring theories. In the hope of learning more about this mysterious theory, it is important to reinterpret the classical solutions to the effective field theory limit of some superstring theories from an eleven-dimensional perspective. The aim of this paper is to consider such interpretation for the Kałuża-Klein electrically charged solutions of type IIA superstring theory.

Ten-dimensional $N = 2A$ supergravity is the effective field theory limit of type IIA ten-dimensional superstring theory. The bosonic fields are the metric g, the 2-form field $\mathcal{B}$ and the dilaton ϕ all arising from the NS-NS sector, and the 1-form field $\mathcal{A}$ and 3-form field $\mathcal{A}_3$ arising from the R-R sector. The 1, 2 and 3-form fields can be used to construct charged black brane solutions of $N = 2A$ supergravity [10]. In particular the 1-form field has been used to construct electrically charged black hole and magnetically charged black 6-brane solutions. The extreme solutions preserve one half of the maximal supersymmetry. These are particularly important because its semi-classical quantisation is believed to be exact [11, 12].

We study Kałuża-Klein electrically charged black p-brane solutions of type IIA superstring theory in both ten and eleven dimensions. In section two all possible cases of this class of solutions are considered. The following section is concerned with the eleven-dimensional picture. In section four we give some concluding remarks. For completeness the solutions for Kałuża-Klein electrically

charged black branes in D dimensions with arbitrary coupling of the dilaton field to the Maxwell field strength are presented in an appendix. Magnetic duality is used to generate magnetically charged black brane solutions.

2 Kałuża-Klein electrically charged black branes

The effective action for the massless background bosonic fields in type IIA superstring theory, obtained by expanding in string loop and sigma model couplings, is the $N = 2A$ supergravity action

$$\begin{aligned} I_{IIA} &= \tfrac{1}{2}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4(\bigtriangledown\phi)^2 - \tfrac{1}{2.3!}\mathcal{H}^2\right) - \tfrac{1}{2.2!}\mathcal{F}^2 - \tfrac{1}{2.4!}\mathcal{F}_4'^2\right] \\ &+ \tfrac{1}{4}\int \mathcal{F}_4\wedge\mathcal{F}_4\wedge\mathcal{B}, \end{aligned} \tag{2.1}$$

where $\mathcal{F}_4' = d\mathcal{A}_3 + \mathcal{A}\wedge\mathcal{H}$, $\mathcal{F}_4 = d\mathcal{A}_3$, $\mathcal{H} = d\mathcal{B}$, $\mathcal{F} = d\mathcal{A}$ and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}_3$ are 1, 2 and 3-form fields respectively. After rescaling the metric $g_{mn} \to e^{\frac{\phi}{2}}g_{mn}$ and setting $\mathcal{B}$ and $\mathcal{A}_3$ equal to zero we have in the Einstein frame

$$I_{IIA} = \frac{1}{2}\int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\bigtriangledown\phi)^2 - \frac{1}{2.2!}e^{\frac{3}{2}\phi}\mathcal{F}^2\right]. \tag{2.2}$$

A class of black p-brane solutions to this action is $(0 \le p \le 6)$

$$\begin{aligned} ds^2 &= \left(1 + \tfrac{\alpha}{r^{7-p}}\right)^{\frac{1}{8}}\left[-\frac{1-\left(\frac{r_H}{r}\right)^{7-p}}{1+\frac{\alpha}{r^{7-p}}}dt^2 + \frac{dr^2}{1-\left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega_{8-p}^2 + dy^s dy_s\right], \\ \phi(r) &= \phi_0 + \ln\left(1 + \tfrac{\alpha}{r^{7-p}}\right)^{\frac{3}{4}}, \\ *\mathcal{F} &= Qe^{-\frac{3}{2}\phi}(\epsilon_{8-p}\wedge\eta_p), \end{aligned} \tag{2.3}$$

where $s = 1, ..., p$ and ϵ_{8-p} and η_p are the volume forms on the unit $(8-p)$-sphere and R_y^p, respectively. The solution for $p = 0$ was first found by Gibbons and Maeda [13] (see also [14]). We note that according to the interpretation given by Papadopoulos and Townsend [15] these objects remain 0-branes after compactifying along the branes spatial dimensions. ϕ_0 is the expectation value of the dilaton field at infinity and determines the string coupling there. The electric charge is defined by $Q = \frac{1}{V_p}\int_\Sigma e^{\frac{3}{2}\phi} * \mathcal{F}$, where $\Sigma = S^{8-p}\times R_y^p$ is an asymptotic spacelike hypersurface and V_p the volume of R_y^p. The ADM mass per unit of p-volume [16] and the electric charge can be written in terms of the constants of integration α and r_H

$$\begin{aligned} \tfrac{M}{A_{8-p}} &= \tfrac{8-p}{2}r_H^{7-p} + \tfrac{7-p}{2}\alpha, \\ \left(\tfrac{Q_0}{(7-p)A_{8-p}}\right)^2 &= \alpha\left(\alpha + r_H^{7-p}\right), \end{aligned} \tag{2.4}$$

where $Q_0 = Qe^{-\frac{3}{4}\phi_0}$. It is possible to define a scalar charge associated with the dilaton field. Noting that the kinetic term of the dilaton field in the action (2.2) is given by $\psi - \psi_0 = \frac{\phi - \phi_0}{2}$, this charge is defined by the asymptotic behaviour of $\psi - \psi_0$

$$\psi - \psi_0 \sim \frac{\Sigma}{(7-p)A_{8-p}} \frac{1}{r^{7-p}}. \tag{2.5}$$

The dilaton field in (2.3) gives

$$\frac{\Sigma}{(7-p)A_{8-p}} = \frac{3}{8}\alpha. \tag{2.6}$$

By changing to null coordinates it is easily seen that $r = r_H$ is a coordinate singularity of the metric in (2.3) (provided that $r_H^{7-p} \neq -\alpha$ and $r_H \neq 0$). The scalar curvature and the Ricci tensor, which are associated with the sources of the gravitational field, as well as the Weyl tensor all diverge for $r = 0$ and $r^{7-p} = -\alpha$ in both the string and Einstein metrics.[1]

2.1 Global Structure

In order to study the global structure we consider all possible values of α and r_H restricted to the physical conditions $M, Q^2 \geq 0$. Inverting (2.4) we have

$$\begin{aligned} \alpha &= \tfrac{M}{A_{8-p}} \left[-1 \pm \sqrt{1 + \tfrac{8-p}{(7-p)^2} \left(\tfrac{Q_0}{M}\right)^2} \right], \\ r_H^{7-p} &= \tfrac{M}{(8-p)A_{8-p}} \left[(9-p) \mp (7-p)\sqrt{1 + \tfrac{8-p}{(7-p)^2} \left(\tfrac{Q_0}{M}\right)^2} \right]. \end{aligned} \tag{2.7}$$

The cases (i) and (ii) below correspond to the upper and lower sign choices, respectively.

(i) $\alpha \geq 0$ and $r_H^{7-p} \leq \frac{2}{8-p}\frac{M}{A_{8-p}}$ $\quad \left(r_H^{7-p} \geq -\frac{7-p}{8-p}\alpha\right)$.

The equality $\alpha = 0$, $r_H^{7-p} = \frac{2}{8-p}\frac{M}{A_{8-p}}$ corresponds to the Schwarzschild black p-brane. For $\frac{2}{8-p}\frac{M}{A_{8-p}} > r_H^{7-p} > 0$ we have $0 < Q_0^2 < 4M^2$ and the Penrose diagram is the same as in the usual four-dimensional Schwarzschild case.

[1]The solution (2.3) is defined for $1 + \frac{\alpha}{r^{7-p}} > 0$. The region $1 + \frac{\alpha}{r^{7-p}} < 0$ could be obtained by taking the modulus of $1 + \frac{\alpha}{r^{7-p}}$. The corresponding background interpolates between the singularities at $r = 0$ and $r^{7-p} = -\alpha$. The $r < 0$ region is not very interesting either.

For $r_H = 0$ we have an extreme solution. The metric in (2.3) becomes

$$ds^2 = \left(1 + \frac{\alpha}{r^{7-p}}\right)^{\frac{1}{8}} \left[-\frac{dt^2}{1 + \frac{\alpha}{r^{7-p}}} + dx^i dx_i + dy^s dy_s\right], \tag{2.8}$$

where $i = 1, ..., 9 - p$ and $r^2 = x^i x_i$. This metric has symmetry $R \times E(p) \times SO(9-p)$, where $E(p)$ is the p-dimensional Euclidean group. According to the terminology followed in [15] these solutions should be interpreted as 0-branes but compactification along the y directions is assumed. Provided that some supersymmetry is left unbroken quantum corrections should vanish and this solution is therefore believed to be exact.

We now check that one half of the maximal supersymmetry is broken. For vanishing fermionic, dilatino, 2 and 3-form background fields, the supersymmetric transformations for the fermionic and dilatino fields in $D = 10$, $N = 2A$ supergravity are

$$\begin{aligned} \delta\psi_m &= D_m\epsilon + \tfrac{1}{64} e^{\frac{3}{4}\phi} \left(\Gamma_m{}^{np} - 14\delta_m^n \Gamma^p\right) \Gamma^{10} \epsilon \mathcal{F}_{np}, \\ \delta\lambda &= -\tfrac{1}{2\sqrt{2}} \left(D_m\phi\right) \Gamma^m \Gamma^{10} \epsilon + \tfrac{3}{16\sqrt{2}} e^{\frac{3}{4}\phi} \Gamma^{mn} \epsilon \mathcal{F}_{mn}, \end{aligned} \tag{2.9}$$

where $\Gamma_{ab...c} = \Gamma_{[a}\Gamma_b ... \Gamma_{c]}$, we use the gamma matrices algebra $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ compatible with a real representation and $\Gamma^{10} = \Gamma^0 ... \Gamma^9$. As usual $m, n, ...$ denote $10D$ world indices and $a, b, ...$ $10D$ tangent space indices. Our field configuration preserves some unbroken supersymmetry if $\delta\psi_m = 0$ and $\delta\lambda = 0$. The Killing spinor that solves this equation is

$$\epsilon = \left(1 + \frac{\alpha}{r^{7-p}}\right)^{-\frac{7}{32}} \epsilon_0, \tag{2.10}$$

with $\Gamma_0 \Gamma^{10} \epsilon_0 = \mp\epsilon_0$ and ϵ_0 a constant spinor. The $\mp$ sign choice corresponds to positive or negative charge , respectively. Thus one half of the maximal supersymmetry is broken. This solution saturates the Bogolmol'nyi bound for the ADM mass per unit of p-volume $2M = |Q_0|$ $[17-19]$. As expected this result does not depend on p as it is derived from the supersymmetry algebra.

To understand the global structure consider the vector v orthogonal to the $r = const.$ hypersurfaces. In components $v^m = g^{mn}\partial_n r$ and $v^2 = \left(1 + \frac{\alpha}{r^{7-p}}\right)^{-1/8}$. Thus $v^2 > 0$ for $r > 0$ but $v^2 \to 0$ for $r \to 0$. This means that the singular hypersurface $r = 0$ is null, i.e. it coincides with the horizon. In fact, $r = 0$ is a Killing horizon in a limiting sense. The Penrose diagram shown in figure 1(a) disagrees with [13], where a diagram corresponding to a naked singularity at $r = 0$ was drawn. Our result is consistent with the proposal of supersymmetry being the cosmic censor [12].

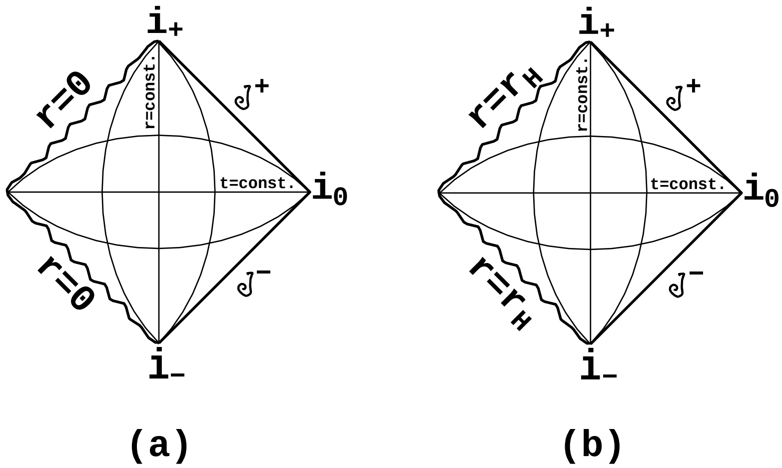

Figure 1: Penrose diagrams for (a) the extreme supersymmetric solution and (b) the uncharged dilatonic black p-brane. Each point is $S^{8-p} \times R^p$.

For $r_H < 0$ we have a naked singularity at $r = 0$.[2]

(ii) $\alpha \leq -\frac{2M}{A_{8-p}}$ and $r_H^{7-p} \geq \frac{2M}{A_{8-p}}$ $\left(-\frac{7-p}{8-p}\alpha \leq r_H^{7-p} \leq -\alpha\right)$.

For $r_H^{7-p} = -\alpha = \frac{2M}{A_{8-p}}$ the metric in (2.3) simplifies to

$$ds^2 = \left(1 - \left(\frac{r_H}{r}\right)^{7-p}\right)^{\frac{1}{8}} \left[-dt^2 + dy^s dy_s + \frac{dr^2}{1 - \left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega_{8-p}^2\right], \tag{2.11}$$

and it has symmetry $P(p+1) \times SO(9-p)$, where $P(p+1)$ is the $(p+1)$-dimensional Poincaré group. This is an uncharged dilatonic black p-brane with the singularity located at the horizon. The Penrose diagram is shown in figure 1(b). A naive calculation of the Hawking temperature based on Euclidean continuation will give a divergent result. Correspondingly, the area of the horizon (calculated along a surface of $t = const.$) is zero. All supersymmetries are broken.

For $\alpha < -\frac{2M}{A_{8-p}}$ and $r_H^{7-p} > \frac{2M}{A_{8-p}}$ we have a naked singularity at $r = (-\alpha)^{1/(7-p)}$.[2]

[2] For $r_H^{7-p} = -\frac{7-p}{8-p}\alpha$ we have a naked singularity with $M = 0$ and $\left(\frac{Q_0}{(7-p)A_{8-p}}\right)^2 = \frac{\alpha^2}{8-p}$.

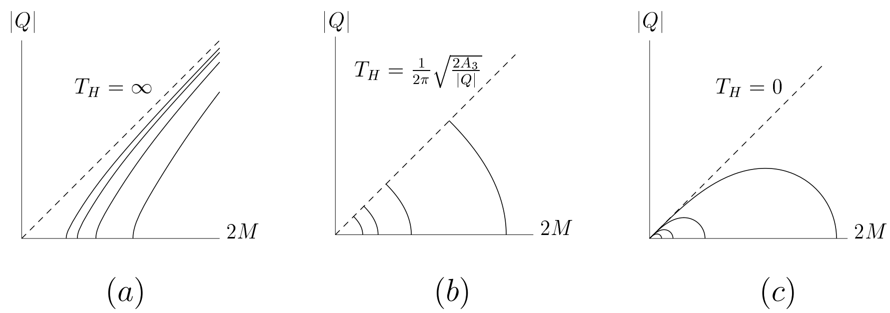

Figure 2: Isothermals in the $Q - M$ plot for (a) $p = 6$, (b) $p = 5$ and (c) $0 \leq p \leq 4$. These cases correspond after dimensional reduction to black holes in $D = 4$, $D = 5$ and $6 \leq D \leq 10$ dimensions, respectively.

2.2 Thermodynamics

The coupling of the dilaton field to the Maxwell field strength, a, determines the global structure and thermal properties. In the previous subsection the solutions of item (i) with $r_H > 0$ represent black p-brane solutions. Therefore the standard thermodynamical arguments should apply.

Analytically continuing to Euclidean spacetime and examining the behaviour of the metric in the vicinity of the horizon $r = r_H$, the Hawking temperature is seen to be

$$T_H = \frac{7-p}{4\pi} r_H^{\frac{5-p}{2}} \left(r_H^{7-p} + \alpha \right)^{-\frac{1}{2}} . \tag{2.12}$$

In the limit $r_H \to 0$ we have

$$T_H \to \frac{7-p}{4\pi} r_H^{\frac{5-p}{2}} \left(\frac{|Q|}{(7-p)A_{8-p}} \right)^{-\frac{1}{2}} . \tag{2.13}$$

For $p = 6$ T_H diverges, for $p = 5$ it converges to a finite number and for $0 \leq p \leq 4$ it converges to zero. In figure 2 the isothermals in the $Q - M$ plot are shown. These plots were presented in [13]. Here we present just those that correspond to the Kałuża-Klein electrically charged black p-branes that have an eleven dimensional origin. In figure 3 we show the $T - Q$ and $T - M$ plots.

The charge of the black p-branes arises as a central charge in a supersymmetric algebra. If quanta of charge cannot be radiated, the evolution due to the Hawking evaporation process is at constant charge. The black p-branes will decrease its mass until the extreme limit is reached while the temperature converges to infinity, a finite value or zero according to $p = 6$, $p = 5$ or $0 \leq p \leq 4$, respectively. In the case of $p = 6$ we expect the thermodynamic description to breakdown near the extremal limit. The case of $p = 5$ is rather puzzling.

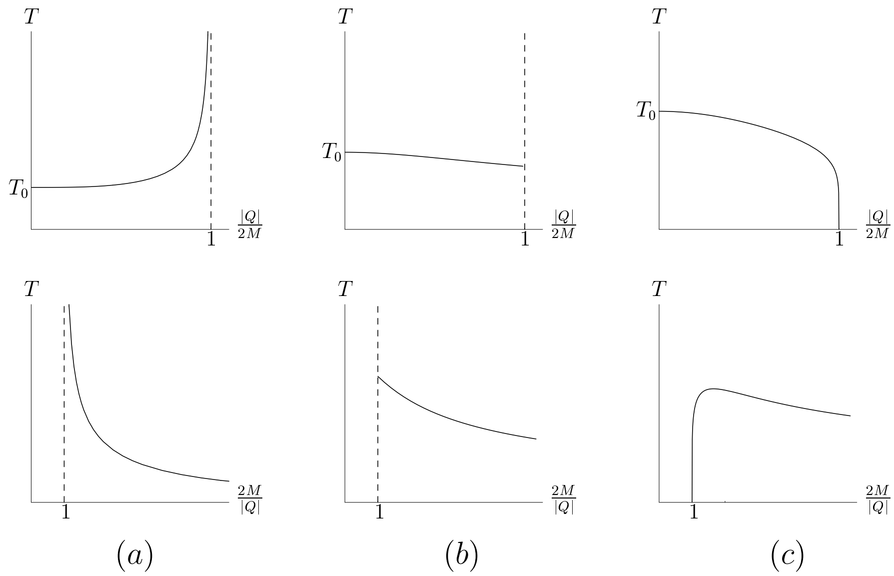

Figure 3: $T-Q$ and $T-M$ plots for (a) $p=6$, (b) $p=5$ and (c) $0\le p\le 4$. T_0 depends on p and is given by $T_0=\frac{7-p}{4\pi}\left(\frac{2M}{(8-p)A_{8-p}}\right)^{-\frac{1}{7-p}}$. The specific heat $c=\frac{\partial M}{\partial T}|_Q$ is negative in cases (a) and (b) but changes sign in case (c) [20].

The equations (2.4) and (2.12) can be used to derive the mass formula

$$M=\frac{8-p}{7-p}T_H\frac{A_H}{4G}+\Phi_H Q, \tag{2.14}$$

where in our units $4G=\frac{1}{2\pi}$, A_H is the horizon area (divided by V_p) and Φ_H is the horizon electric potential. The differential form of (2.14) is the first law of thermodynamics

$$dM=T_H\frac{dA_H}{4G}+\Phi_H dQ. \tag{2.15}$$

The macroscopic specific entropy can be read from this equation and it is given by

$$S_{BH}=2\pi A_{8-p}{r_H}^{\frac{9-p}{2}}\left(r_H^{7-p}+\alpha\right)^{\frac{1}{2}}. \tag{2.16}$$

3 Eleven-dimensional Spacetime

Dimensional reduction of $D=11$ supergravity on $M\times S^1$ yields $N=2$, non-chiral, ten-dimensional supergravity whose bosonic sector action is given by

(2.1) [21]. The action for the bosonic sector of eleven dimensional supergravity is

$$I_{11} = \frac{1}{2}\int d^{11}x\sqrt{-g}\left[R - \frac{1}{2.4!}\mathcal{F}_4^2\right] + \frac{1}{12}\int \mathcal{F}_4 \wedge \mathcal{F}_4 \wedge \mathcal{A}_3, \qquad (3.1)$$

where $\mathcal{F}_4 = d\mathcal{A}_3$ and $\mathcal{A}_3$ is a 3-form field. Dimensional reduction to the Einstein frame is performed by writing

$$E_M^A = \begin{pmatrix} e^{-\frac{1}{12}\phi}e_m^a & -e^{\frac{2}{3}\phi}\mathcal{A}_m \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix}$$
$$\mathcal{A}_{MNP} = \mathcal{A}_{mnp},\ \mathcal{A}_{MN10} = \mathcal{B}_{mn}, \quad with\ M,N,P = 0,...,9. \qquad (3.2)$$

Unless stated capital letters range from 0 to 10. $M, N, ...$ denote $11D$ world indices and $A, B, ...$ $11D$ tangent space indices.

The eleven dimensional metric corresponding to the solution (2.3) is obtained in the following way

$$ds^2 = g_{MN}dx^M dx^N = e^{-\frac{1}{6}\phi}g_{mn}dx^m dx^n + e^{\frac{4}{3}\phi}\left(dx^{10} - \mathcal{A}_m dx^m\right)^2, \qquad (3.3)$$

$$\begin{aligned} ds^2 = & -\left(1 - \tfrac{r_H^{7-p}+\alpha}{r^{7-p}}\right)dt^2 + \left(1 + \tfrac{\alpha}{r^{7-p}}\right){dx^{10}}^2 \mp 2\tfrac{\sqrt{\alpha\left(r_H^{7-p}+\alpha\right)}}{r^{7-p}}dt dx^{10} \\ & + \tfrac{dr^2}{1-\left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega_{8-p}^2 + dy^s dy_s, \end{aligned} \qquad (3.4)$$

where the $\mp$ sign choice corresponds to positive or negative charge, respectively.[3] Going from (3.3) to (3.4) we performed the rescalings $x^m \to e^{\frac{1}{12}\phi_0}x^m$ and $x^{10} \to e^{-\frac{2}{3}\phi_0}x^{10}$.

The special cases $r_H = 0$ and $r_H^{7-p} = -\alpha$ will be treated separately afterwards. If $\alpha \geq 0$ and $r_H > 0$ it can be seen that $r = r_H$ is a Killing horizon of $\frac{\partial}{\partial w} = \frac{\partial}{\partial t} \pm \left(\frac{\alpha}{r_H^{7-p}+\alpha}\right)^{1/2}\frac{\partial}{\partial x^{10}}$. This suggests that the metric (3.4) can be written in the coordinates $w(t, x^{10})$ and $y(t, x^{10})$ with $g\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial y}\right) = 0$. The transformation that does the job is

$$\begin{aligned} w &= \left(1 + \tfrac{\alpha}{r_H^{7-p}}\right)t \mp \tfrac{\sqrt{\alpha\left(r_H^{7-p}+\alpha\right)}}{r_H^{7-p}}x^{10}, \\ y &= \mp\tfrac{\alpha}{r_H^{7-p}}t + \tfrac{\sqrt{\alpha\left(r_H^{7-p}+\alpha\right)}}{r_H^{7-p}}x^{10}, \end{aligned} \qquad (3.5)$$

and the result is

$$ds^2 = -\frac{1-\left(\frac{r_H}{r}\right)^{7-p}}{1+\frac{\alpha}{r_H^{7-p}}}dw^2 + \frac{r_H^{7-p}}{\alpha}dy^2 + \frac{dr^2}{1-\left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega_{8-p}^2 + dy^s dy_s. \qquad (3.6)$$

[3]In the region $1 + \frac{\alpha}{r^{7-p}} < 0$ the first three terms in the metric have their signs changed (see footnote 1).

After a rescaling of w and y we can see that α does not play any role in the eleven-dimensional metric. From a $11D$ point of view it is just introduced by writing the metric in the form (3.6) and changing to the t, x^{10} coordinates to obtain the straightforward compactifiable metric (3.4). We now consider the different possibilities:

(i) $\alpha \geq 0$, $r_H^{7-p} \leq \frac{2}{8-p}\frac{M}{A_{8-p}}$ and $r_H \neq 0$ $\quad \left(r_H^{7-p} \geq -\frac{7-p}{8-p}\alpha\right)$.

Consider first $r_H > 0$. Performing the rescalings $\sqrt{\frac{r_H^{7-p}}{r_H^{7-p}+\alpha}}w \to w$ and $\sqrt{\frac{r_H^{7-p}}{\alpha}}y \to y$ we obtain the Schwarzschild black $(p+1)$-brane metric

$$ds^2 = -\left[1-\left(\frac{r_H}{r}\right)^{7-p}\right]dw^2 + \frac{dr^2}{1-\left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega_{8-p}^2 + dy^2 + dy^s dy_s. \quad (3.7)$$

The coordinate transformation (3.5) plus the w and y rescalings can be seen to generate a boost along the compact direction with velocity given by $v^2 = \frac{\alpha}{r_H^{7-p}+\alpha}$. The Schwarzschild case $\alpha = 0$, $r_H^{7-p} = \frac{2}{8-p}\frac{M}{A_{8-p}}$ is obtained by setting $w = t$ and $y = x^{10}$ in (3.7) (it corresponds to zero boost velocity, i.e. no coordinate transformation).

For $r_H < 0$ we perform the rescalings $\sqrt{\frac{-r_H^{7-p}}{r_H^{7-p}+\alpha}}w \to w$ and $\sqrt{\frac{-r_H^{7-p}}{\alpha}}y \to y$ and (3.6) becomes

$$ds^2 = -dy^2 + dy^s dy_s + \left[1-\left(\frac{r_H}{r}\right)^{7-p}\right]dw^2 + \frac{dr^2}{1-\left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega_{8-p}^2. \quad (3.8)$$

Now, the y direction is timelike and the w direction spacelike. (3.5) plus the rescalings generate a boost along the compact direction with velocity given by $v^2 = \frac{r_H^{7-p}+\alpha}{\alpha}$. Because $r_H < 0$ there is a naked singularity at $r = 0$.

(ii) $\alpha < -\frac{2M}{A_{8-p}}$ and $r_H^7 > \frac{2M}{A_{8-p}}$ $\quad \left(-\frac{7-p}{8-p}\alpha \leq r_H^{7-p} < -\alpha\right)$.

Performing the rescalings $\sqrt{\frac{r_H^{7-p}}{-(r_H^{7-p}+\alpha)}}w \to w$ and $\sqrt{\frac{r_H^{7-p}}{-\alpha}}y \to y$ we obtain the metric (3.8) but now r_H is positive. As before the y direction is timelike, the w direction spacelike and the boost velocity given by $v^2 = \frac{-(r_H^{7-p}+\alpha)}{-\alpha}$. The metric (3.8) (for $p = 0$) has been identified as a vacuum solution of $11D$ supergravity [4]. In the region $r_H < r < (-\alpha)^{1/(7-p)}$, $m = \frac{\partial}{\partial x^{10}}$ becomes

timelike and our compactification scheme breaks down as can be seen in (3.3). This region is absent in the solution (2.3).[4]

Consider now the extreme cases $r_H^{7-p} = -\alpha$ and $r_H = 0$:

(iii) $r_H^{7-p} = -\alpha \quad (\alpha < 0)$.

The metric (3.4) becomes

$$ds^2 = -dt^2 + dy^s dy_s + \left[1 - \left(\frac{r_H}{r}\right)^{7-p}\right] dx^{10^2} + \frac{dr^2}{1 - \left(\frac{r_H}{r}\right)^{7-p}} + r^2 d\Omega^2_{8-p}. \quad (3.9)$$

This metric is similar to the one in (ii) if we replace t by y and x^{10} by w (it corresponds to zero boost velocity, i.e. no coordinate transformation). The spacetime is the product of the $(10-p)$-dimensional Euclidean Schwarzschild manifold with the $(p+1)$-dimensional Minkowski spacetime. If x^{10} has period $\frac{4\pi}{7-p} r_H$ we can avoid a conical singularity at $r = r_H$ and our spacetime has a topology $R \times R^2 \times S^{8-p} \times R^p$. This solution has $P(p+1) \times SO(9-p) \times SO(2)$ symmetry. x^{10} is naturally periodic avoiding the singularity at $r = r_H$ that was present in the ten dimensional version. At $r = r_H$ the compactification radius vanishes in agreement with the fact that in the ten-dimensional solution the dilaton field was $-\infty$ there. While in the latter the Hawking temperature was infinity and the horizon area vanished, now $T = 0$ because the time direction is flat and $A(r = r_H) = r_H^{8-p} A_{8-p} V_p$.

(iv) $r_H = 0 \quad (\alpha > 0)$.

The metric (3.4) simplifies to

$$ds^2 = -du dv + \frac{\alpha}{r^{7-p}} du^2 + dx^i dx_i + dy^s dy_s, \quad (3.10)$$

with $i = 1, ..., 9-p$ and $v, u = t \pm x^{10}$ ($du^2 \to dv^2$ for negative charge). This is a plane wave metric [22], i.e. $\partial_T^2 \left(\frac{\alpha}{r^{7-p}}\right) = 0$ $(r \neq 0)$, where ∂_T^2 is the Laplacian in the transverse x^i space. This solution has $SO(9-p) \times E(p)$ symmetry.

For vanishing fermionic and 3-form field backgrounds the supersymmetric transformation for the fermionic field in eleven dimensional supergravity

[4]To obtain the region $1 + \frac{\alpha}{r^{7-p}} < 0$ we should have started with the $10D$ metric defined in this region (see footnotes 1 and 3).

is simply $\delta\psi_M = D_M\epsilon$. The Killing spinor field solving $\delta\psi_M = 0$ can be related to $\epsilon^{(10)}$ by

$$\epsilon^{(11)} = \left(1 + \frac{\alpha}{r^{7-p}}\right)^{-\frac{1}{4}} \epsilon_0 = \left(1 + \frac{\alpha}{r^{7-p}}\right)^{-\frac{1}{32}} \epsilon^{(10)}, \tag{3.11}$$

with $\Gamma_0\Gamma_{10}\epsilon_0 = \mp\epsilon_0$, ϵ_0 a constant spinor and the gamma matrices algebra is now $\{\Gamma_A, \Gamma_B\} = \eta_{AB}$.

The Ricci tensor for this solution can be written distributionally as

$$R_{MN} = -2\partial_T^2 \left(\frac{\alpha}{r^{7-p}}\right) U_M U_N = 2(7-p)\alpha A_{8-p}\delta^{9-p}(x^i) U_M U_N, \tag{3.12}$$

with $U = \frac{\partial}{\partial v}$. Making use of Einstein equations the energy-momentum tensor is

$$T^{MN} = 2(7-p)\alpha A_{8-p}\delta^{9-p}(x^i) U^M U^N. \tag{3.13}$$

This can be interpreted as a $p+1$ spatially extended object (or a string according to [15]) sited at the origin with a 11-velocity $U = \frac{\partial}{\partial v}$. It is trapped around the eleventh dimension along which momentum is flowing. Periodicity on S^1 of the wave function yields quantisation of this momentum and therefore of the KK charge.

We could have done a similar analysis for the region $1 + \frac{\alpha}{r^{7-p}} < 0$ and/or $r < 0$ but the results are less interesting. This exhausts all possible cases.

4 Conclusion

From our results we can conclude that the class of ten-dimensional black p-branes described by (2.3) when uplifted to eleven dimensions on $M \times S^1$ have the following interpretation:

(i) $\alpha \geq 0$, $r_H^{7-p} \leq \frac{2}{8-p}\frac{M}{A_{8-p}}$ and $r_H \neq 0$. If $r_H > 0$ we have a boosted Schwarzschild black $(p+1)$-brane. The boosted and compactified direction is one of the brane's spatial directions. The ten-dimensional Schwarzschild p-brane is obtained when there is no boost. If $r_H < 0$ we have a naked singularity.

(ii) $\alpha < -\frac{2M}{A_{8-p}}$, $r_H^{7-p} > \frac{2M}{A_{8-p}}$. The background is given by the boost of the product of the $(10-p)$-dimensional Euclidean Schwarzschild manifold with the $(p+1)$-dimensional Minkowski spacetime. The boosted and compactified direction is the Euclidean Schwarzschild time direction. The compactification scheme breaks down for $r^{7-p} < -\alpha$ where $\frac{\partial}{\partial x^{10}}$ becomes timelike. The corresponding ten-dimensional background has a naked singularity at $r^{7-p} = -\alpha$ where $\phi \to -\infty$, i.e. the string coupling vanish.

(iii) $\alpha = -\frac{2M}{A_{8-p}} = -r_H^{7-p}$. Same background as in (ii) but without any boost. Provided that x^{10} has period $\frac{4\pi}{7-p} r_H$ the hypersurface $r = r_H$ is not singular. It corresponds in the ten-dimensional version to a null singularity where the string coupling vanish.

(iv) $\alpha > 0$, $r_H = 0$. The solution corresponds to a $p+1$ spatially extended object trapped around the eleventh dimension along which momentum is flowing.

The latter is supersymmetric and its semiclassical quantisation is therefore believed to be exact. This is consistent with one of the key ideas of M-theory, namely solitons of string theory are elementary excitations of M-theory.

Although these results were derived in the context of M-theory we expect them to be generally valid for Kałuża-Klein electrically charged black branes in any spacetime dimension.

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Appendix

In this paper we considered Kałuża-Klein electrically charged black branes in type IIA superstring theory. For completeness we will present this solutions for the more general action

$$I = \frac{1}{2}\int d^D x \sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2.2!}e^{a\phi}\mathcal{F}^2\right]. \tag{A.1}$$

The electrically charged black p-brane solutions are given by $(0 \le p \le D-4)$

$$\begin{aligned} ds^2 &= {F_\alpha}^\delta\left[-\frac{F_H}{{F_\alpha}^{\delta(D-2)}}dt^2 + \frac{dr^2}{F_H} + r^2 d\Omega^2_{D-2-p} + dy^s dy_s\right], \\ \phi &= \phi_0 + \frac{D-2}{2} a \ln {F_\alpha}^\delta, \\ *\mathcal{F} &= Q e^{-a\phi}(\epsilon_{D-2-p} \wedge \eta_p), \end{aligned} \tag{A.2}$$

where $F_\alpha = 1 + \frac{\alpha}{r^{D-3-p}}$, $F_H = 1 - \left(\frac{r_H}{r}\right)^{D-3-p}$, $s = 1, ..., p$ and $2\delta^{-1} = D - 3 + \frac{D-2}{2}a^2$. The electric charge is defined by $Q = \frac{1}{V_p}\int_\Sigma e^{a\phi} * \mathcal{F}$, where $\Sigma = S^{D-2-p} \times R^p_y$ is

an asymptotic spacelike hypersurface and V_p the volume of R_y^p. The ADM mass per unit of p-volume and the charge Q are given by

$$\begin{aligned}
\tfrac{2M}{(D-3-p)A_{D-2-p}} &= \tfrac{D-2-p}{D-3-p} r_H^{D-3-p} + (D-2)\delta\alpha, \\
\left(\tfrac{Q_0}{(D-3-p)A_{D-2-p}}\right)^2 &= (D-2)\delta\alpha\left(\alpha + r_H^{D-3-p}\right),
\end{aligned} \tag{A.3}$$

where $Q_0 = Qe^{-\frac{a}{2}\phi_0}$. The Hawking temperature is

$$T_H = \frac{D-3-p}{4\pi}\frac{1}{r_H\left(1+\frac{\alpha}{r_H^{D-3-p}}\right)^{(D-2)\frac{\delta}{2}}}, \tag{A.4}$$

and the mass formula

$$M = \frac{D-2-p}{D-3-p}T_H\frac{A_H}{4G} + \Phi_H Q, \tag{A.5}$$

where Φ_H is the horizon electric potential. The first law of thermodynamics is still given by

$$dM = T_H\frac{dA_H}{4G} + \Phi_H dQ, \tag{A.6}$$

and the macroscopic specific entropy is

$$S = 2\pi\left(1+\frac{\alpha}{r_H^{D-3-p}}\right)^{(D-2)\frac{\delta}{2}} r_H^{D-2-p}A_{D-2-p}. \tag{A.7}$$

Magnetic duality can be used to obtain magnetically charged black p-brane solutions with the charge arising from a $(D-2)$-form field strength. Let $\tilde{\mathcal{F}} = e^{a\phi} * \mathcal{F} = Q(\epsilon_{D-2-p}\wedge\eta_p)$. If $\mathcal{F}$ is replaced by $\tilde{\mathcal{F}}$ in the equations of motion that follow from (A.1) we obtain a set of equations that can be derived from the action

$$\tilde{I} = \frac{1}{2}\int d^Dx\sqrt{-g}\left[R - \frac{1}{2}(\bigtriangledown\phi)^2 - \frac{1}{2.(D-2)!}e^{-a\phi}\tilde{\mathcal{F}}^2\right]. \tag{A.8}$$

Thus the magnetically charged black p-brane solutions are obtained from the previous solutions by making the replacement $a \to -a$. The magnetic charge is given by $Q = \frac{1}{V_p}\int_\Sigma \tilde{\mathcal{F}}$, with $\Sigma = S^{D-2-p}\times R_y^p$ an asymptotic spacelike hypersurface.

In the cases where the actions I or Ĩ describe part of the bosonic fields content of some supergravity theory we expect the extreme solutions $r_H = 0$ to be supersymmetric and to saturate the Bogolmol'nyi bound for the ADM mass per unit of p-volume $2M = \sqrt{\delta(D-2)}|Q_0|$.

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